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Note on constructal theory of organization in nature

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Abstract

In analyses of engineering systems, fundamental quest remains the distribution of certain entities (matter, energy) into the space. Thus certain definite forms are generated to serve a specific purpose. Bejan's constructal theory specifically deals with such optimal geometrical constructions. In this present article, equipartition theory is revisited from macroscopic standpoint. It is noted that equipartition principle is a corollary of a more generalized formulation—the constructal theory. It is seen that equipartition principle leads to certain power laws with which certain entities are distributed. This constitutes a deterministic law of nature in some finite length and time scale. Thus equipartition principle is to be recognized as an authentic basis of design for most efficient systems, which in turn obeys constructal principle of organization in nature.

Keywords: Constructal; Equipartition; Flow; Non-equilibrium systems; Self-organization

1. Constructal theory of organization

Recent advancement in thermodynamic optimization is presented here with reference to the generation of optimal geometric forms (topology) in flow systems. The flow configuration has flexibility to alter its shape and structure. The motive that governs the generation of geometric form is in the fulfillment of minimum flow resistance criterion. The imposed constraint is global finiteness: volumetric flow rate, weight of the fluid and time rate of flow. The emerging structures obtained in this manner are termed constructal designs. The same objective and constraints resulting in the similar structure that accommodates optimally shaped flow paths occurring in nature and artificial systems are named constructal law [1–7]. It is the single theory encompassing the observations covered in animate and inanimate systems. The law can be stated as follows [1]:

"For a finite-size system to persist in time (to live), it must evolve in such a way that it provides easier access to the imposed currents that flows through it".

It has long been observed that many of the volume-to-point and point-to-volume flows occurring in nature are in the form of tree networks [8–10]. The urge for formulating physics based theory for the generating mechanism, from which fractal-like (not actually fractal but purely Euclidean) structure [1,11] could be predicted, was first met by the constructal theory of volume-to-point flows [2–7].

This theory gave birth out of engineering optimization of paths of minimum thermal resistance for cooling finite-size small-scale electronic components [2]. The problem was to cool a finite-size volume by pure conduction. The statement of this fundamental problem is:

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Nomenclature

A	area
F	force
g	gravitational acceleration
H	total length of fluid stream
h	elemental length of fluid stream
I	area moment of inertia
n	index of power law
p	pressure
Q	volume of fluid stream
R	universal gas constant
T	temperature
V	velocity of fluid stream
v	specific volume of the fluid
w	width of the finite fluid stream

Greek symbols

α	inclination of a fluid element with the horizontal
γ	specific weight of fluid element
δ	elemental length
θ	included angle between two non-parallel sides of a fluid element
ρ	density of fluid element
Δ	infinitesimal potential difference

Subscripts

C	centroid
i	number of segments considered in a continuous fluid stream
n	integer number of partition considered in a finite length of fluid element
P	reference pole
x	flow direction

Superscript

–	location of hydrostatic force

Consider a finite-size volume in which heat is being generated at every point and which is cooled through a small patch (heat sink) located on its boundary. A finite amount of high conductivity material is available. Determine the optimal distribution of such high conductivity material through the given volume such the highest temperature is minimized [12]. The predicted structure reveals a manifestation of the principle of equipartition: the temperature drop through the high conductivity insert is equal to the temperature drop through low conductivity matrix [13]. The second important feature of this optimum can be recorded from the expression of minimized maximum temperature difference ΔT, which scales with the square of orthogonal dimension H of the heat conducting volume to the direction of applied heat current [13] i.e.,

$$\Delta T \sim H^2. \tag{1}$$

Thus, a power law correlates temperature drop and lateral dimension of the cooled volume. The message is to manufacture smallest possible elemental system.

In another realistic access optimization problem we arrive in a situation of point-to-volume flow. The statement of this fundamental problem is as follows: Consider a fluid network to bathe a finite-size volume. The function of the path network is to distribute a stream of fluid to every elemental volume of the space. The mass flow rate of the fluid is purely due to pressure gradient (Hagen–Poiseuille) of the flow. The pressure differential varies with the position of the elemental volume relative to the point source. The maximum pressure difference, which is demanded by the elemental volumes that are situated furthest from the source, is of specific importance. The total mass flow rate is fixed. The thermodynamic optimization of this fluid network is equivalent to minimizing the maximum pressure difference [14]. Optimized result yields that minimized maximum pressure drop ΔP scales with the square of the orthogonal dimension of the bathed volume to the direction of applied fluid flow [15] i.e.,

$$\Delta P \sim H^2. \tag{2}$$

The lesson of this power law correlation is to construct narrowest possible elemental system. If the bifurcation of each path is assumed, each path width shrinks by a factor $\frac{1}{2}$ from one stage to the next smaller stage [14]. Once again principle of equipartition is the crucial underlying feature.

The objective of this present article is to discuss point-to-point (we termed as elemental Fermat type) flow and volume-to-volume (we named as integral Fermat type) flow situations with reference to a fluid system. The results can be extended to a heat transporting system. The analogy and similarity between heat current and fluid stream is theoretically well established [16]. Here results obtained are supplementary to the views expressed in the Ref. [17]. The contrast between Fermat principle and constructal law has been enunciated with clarity by Bejan [18].

2. Elemental Fermat type flow

For a large number of classes of naturally organized (self-organized) systems it is important to establish the effect of gravitation on the thermodynamic properties of the systems. First, it is instructive to establish the distribution of pressure p and specific volume v along the height h of a stack of fluid column. Then, we must use an empirical equation of state for the given substance in the functional form $v = v(p)$ else we must use the method of successive approximations for which we need either experimental data or values calculated via the equation of state both relating to the p versus v dependence along the specific isotherm for the substance studied.

From the basic hydrostatic law it is known that in a column of fluid the pressure varies with height. The change in pressure along the elementary column of height dh is

$$\mathrm{d}p = -\frac{\gamma}{A}\,\mathrm{d}V, \tag{3}$$

where γ is the specific weight of the fluid in the column, dV is the elementary volume and A is the cross-sectional area of the elementary column. Since $\mathrm{d}V = A\mathrm{d}h$, Eq. (3) reduces to

$$\mathrm{d}p = -\gamma\,\mathrm{d}h. \tag{4}$$

By definition, $\gamma = \frac{g}{v}$, where g is the gravitational acceleration. Thus, we arrive at the following equation

$$\mathrm{d}p = -\frac{g}{v}\,\mathrm{d}h, \tag{5}$$

where the minus sign shows that with the increasing height ($\mathrm{d}h > 0$) the fluid pressure decreases ($\mathrm{d}p < 0$). With the choice of reference frame at the top of the free surface instead of the bottom, this sign convention is reversed.

If the pressure p and temperature T of the gas are such that the fluid can be regarded as ideal, the equation of state translates into

$$v = \frac{RT}{p}, \tag{6}$$

where R is the universal gas constant. In view of Eq. (6), we can rewrite Eq. (5) as

$$\mathrm{d}p = -\frac{p}{RT}\,\mathrm{d}h, \tag{7}$$

whence

$$\frac{\mathrm{d}p}{p} = -\frac{\mathrm{d}h}{RT}. \tag{8}$$

Integrating this equation with respect to a reference pressure p_1 at a reference height h_1 we obtain

$$\ln\frac{p(h)}{p_1} = -\frac{1}{R}\int_{h_1}^{h}\frac{\mathrm{d}h}{T}. \tag{9}$$

For isothermal fluid column we have

$$\ln\frac{p(h)}{p_1} = -\frac{h-h_1}{RT}. \tag{10}$$

Thus, we obtain the following formula for the distribution of pressure in an ideal gas isothermal column, known as barometric height formula

$$p(h) = p_1\exp\left(-\frac{h-h_1}{RT}\right). \tag{11}$$

Invoking the ideal gas law, Eq. (6) into Eq. (11) we find that

$$v(h) = v_1\exp\left(\frac{h-h_1}{RT}\right). \tag{12}$$

Hence, we see from the last but one relationship, Eq. (11), dependence of pressure on height of the fluid column is of exponential nature. For small argument of the exponent, the relationship is almost linear.

In view of constructal theory in a self-organized system, certain entities are equipartitioned. For a point-to-point flow configuration, distribution of pressure is of concern. We are interested to learn how the height H of an isothermal vertical fluid column can be divided into n horizontal parts so that pressure is equal in each subdivision.

Let, w be the width of the fluid column. Suppose, below the top of the fluid column, h_1 and h_2 be the depths of the two horizontal lines, which divide the column into three portions. Say p_1, p_2 and p_3 be the three pressures respectively from the surface of fluid on the three portions of the column. The expressions for pressure can be written as follows

$$p_1 = \frac{1}{2}\gamma w h_1^2, \tag{13}$$

$$p_2 = \frac{1}{2}\gamma w (h_2^2 - h_1^2) \tag{14}$$

and

$$p_3 = \frac{1}{2}\gamma w (H^2 - h_2^2). \tag{15}$$

Now, we impose the condition

$$p_1 = p_2 = p_3. \tag{16}$$

Eliminating the pressure terms from Eqs. (13) and (14) we have

$$h_1 = \left(\frac{1}{2}\right)^{1/2} h_2. \tag{17}$$

Again, eliminating the pressure terms between Eqs. (14) and (15) we arrive at

$$h_2 = \left(\frac{1}{2}\right)^{1/2} (h_1^2 + H^2)^{1/2}. \tag{18}$$

Solving Eqs. (17) and (18) for h_1 and h_2 in terms of H we obtain

$$h_1 = \left(\frac{1}{3}\right)^{1/2} H \tag{19}$$

and

$$h_2 = \left(\frac{2}{3}\right)^{1/2} H. \tag{20}$$

On following the method of induction, in general, we can write

$$h_i = \left(\frac{i}{n}\right)^{1/2} H \tag{21}$$

for $i = 1, 2, 3, \ldots, n$.

The center of pressure can be determined to find the coordinate of a representative pressure differential, as the role played by center of mass in solid mechanics in place of a rigid body. Let, $\bar{h}_1$, $\bar{h}_2$ and $\bar{h}_3$ be the depth of center of pressures below the top surface of the fluid column for the three portions of the column. The location of hydrostatic force $\bar{h}_P$ with respect to some pole P is related to the location of hydrostatic force $\bar{h}$ with reference to centroid C by the parallel-axis theorem [19] as follows

$$\bar{h}_P = \bar{h} + I_C \frac{\mathrm{Sin}^2\alpha}{A\bar{h}}, \tag{22}$$

where I_C is the moment of inertia with respect to the centroid and α is the inclination of the fluid column with the horizontal. Here, in particular

$$\alpha = \frac{\pi}{2}, \quad I_C = \frac{1}{12} wh^3, \quad A = wh \quad \text{and} \quad \bar{h} = \frac{h}{2}.$$

Thus for the first partition from the top we have

$$\bar{h}_1 = \frac{2}{3} h_1. \tag{23}$$

Substituting back the value from Eq. (19) to Eq. (23) we arrive at

$$\bar{h}_1 = \frac{2}{3}\left(\frac{1}{3}\right)^{1/2} H. \tag{24}$$

Similarly, for the second partition from the top we get

$$\bar{h}_2 = \frac{2}{3}\left(\frac{2^{3/2} - 1}{3^{1/2}}\right) H. \tag{25}$$

And for the third portion from the top we obtain

$$\bar{h}_3 = \frac{2}{3}\left(\frac{3^{3/2} - 2^{3/2}}{3^{1/2}}\right) H. \tag{26}$$

Thus, generalizing the result on following the method of induction, we finally arrive at

$$\bar{h}_i = \frac{2}{3}\left(\frac{i^{3/2} - (i-1)^{3/2}}{n^{1/2}}\right) H \tag{27}$$

for $i = 1, 2, 3, \ldots, n$.

Contrary to the equipartition of the physical quantity pressure, now we would like to consider equipartition of space and then the distribution of pressure there in. Let, the fluid column H be divided into n large number of equal-sized slices, such that

$$h = \frac{H}{n}. \tag{28}$$

Suppose, the densities of these layers be ρ_1, ρ_2, $\rho_3, \ldots$ and ρ_n respectively. These densities are practically constant over these small slices. Obeying ideal gas equation of state, their corresponding pressures are $RT\rho_1$, $RT\rho_2$, $RT\rho_3, \ldots$ and $RT\rho_n$ respectively. Since, the size of the slices are small, the same pressure is valid at all points of the slice. It means that center of pressure is of no specific importance here. Again, in infinitesimal sense, the difference of pressures on the top and bottom faces of a slice is equal to the weight of the fluid contained in the layer. Hence, we can write in succession

$$RT\rho_1 - RT\rho_2 = \rho_1 gh \tag{29a}$$

$$RT\rho_2 - RT\rho_3 = \rho_2 gh \tag{29b}$$

....

and

$$RT\rho_{n-1} - RT\rho_n = \rho_{n-1} gh \tag{29c}$$

From Eq. (29a) we get

$$\rho_2 = \rho_1\left(1 - \frac{gh}{RT}\right). \tag{30a}$$

Similarly from Eq. (29b), using the result of Eq. (29a) we obtain

$$\rho_3 = \rho_2\left(1 - \frac{gh}{RT}\right) = \rho_1\left(1 - \frac{gh}{RT}\right)^2. \tag{30b}$$

Thus in general we can write

$$\rho_n = \rho_{n-1}\left(1 - \frac{gh}{RT}\right) = \rho_1\left(1 - \frac{gh}{RT}\right)^{n-1}. \tag{30c}$$

Hence, as the altitude increases in arithmetic progression, the densities and the corresponding pressures decrease in geometric progression from the bottom of the vertical fluid column. Now, if ρ be the density just above the nth layer, from Eq. (30c) we have

$$\rho = \rho_n\left(1 - \frac{gh}{RT}\right) = \rho_1\left(1 - \frac{gh}{RT}\right)^n. \tag{30d}$$

Invoking Eq. (28) into Eq. (30d) and rewriting we arrive at the expression below

$$\rho = \rho_1\left[\left(1 - \frac{1}{z}\right)^{-z}\right]^{\left(-\frac{gH}{RT}\right)}, \tag{30e}$$

where

$$z = \frac{nRT}{gH}.$$

For $n \to \infty$, H remains constant and $z \to \infty$. Recognizing the limit

$$\lim_{z\to\infty}\left(1-\frac{1}{z}\right)^{-z} = e, \tag{30f}$$

we finally have

$$\rho = \rho_1 \exp\left(-\frac{gH}{RT}\right). \tag{30g}$$

Following ideal gas law, the expression for pressure takes on the following form

$$p = p_1 \exp\left(-\frac{gH}{RT}\right). \tag{30h}$$

As expected, this last equation is identically the same as that of Eq. (11) obtained earlier. The message of the foregoing analysis is that equipartition of one entity demands the power law distribution of the other associated with it.

Next, we may be interested to learn the range of values of the index of the power law distribution of a physical quantity for which the equipartition of other quantity is valid. In the following example we consider the gauge pressure distribution on the face of a vertical rectangular sluice gate in a free surface flow. From the experimental evidence, the gauge pressure distribution conforms to a mathematical relation of the form [20]

$$p - p_{\text{atm}} = \rho g h\left[1-\left(\frac{h}{H}\right)^n\right], \tag{31}$$

where p_{atm} is the atmospheric pressure exerted on the free surface of the flow, H is the depth of the gate and n is some parametric constant. We are interested to estimate the magnitude and location of the resulting horizontal force on the gate.

Elemental pressure force $\mathrm{d}F_x$ in the horizontal direction on an elemental strip of width w and height $\mathrm{d}h$ is

$$\mathrm{d}F_x = (p - p_{\text{atm}})w\,\mathrm{d}h. \tag{32}$$

Using Eq. (31) into Eq. (32), we get

$$\mathrm{d}F_x = \rho g w\left(h - \frac{h^{n+1}}{H^n}\right)\mathrm{d}h. \tag{33}$$

Total horizontal force is obtained upon integrating Eq. (33) as

$$F_x = \frac{1}{2}\rho g w H^2\left(\frac{n}{n+2}\right). \tag{34}$$

Employing the concept of averaging, we calculate the location h_P of hydrostatic force as

$$h_p F_x = \int_0^H h\,\mathrm{d}F_x. \tag{35}$$

Substituting the expressions for $\mathrm{d}F_x$ and F_x from Eqs. (33) and (34) respectively into Eq. (35) we get

$$h_p = \frac{2}{3}H\left(\frac{n+2}{n+3}\right). \tag{36}$$

On passing to the limit $n \to \infty$ we obtain

$$\lim_{n\to\infty}(F_x) = \frac{1}{2}\rho g w H^2 \tag{37}$$

and

$$\lim_{n\to\infty}(h_p) = \frac{2}{3}H. \tag{38}$$

Thus Eqs. (37) and (38) are asymptotic to the usual results when the index of the power law is very great.

3. Integral Fermat type flow

It is suggestive to become curious about the happenings around us to learn the functioning mechanism of nature. An example of such cadre is the hydraulic jump, which often takes place in the study of river morphology. It is a sudden discontinuity in the depth of the flowing fluid. During the period of tide, a jump may sometimes be observed by standing or moving upstream. This phenomenon of jump can easily be reproduced in laboratory scale. A plate held horizontally under the faucet of fluid may be employed to demonstrate a hydraulic jump. The moving fluid is allowed to hit the center of the plate. Then the fluid flows radially outward in the form of fast thin layer and suddenly increases in thickness before flowing over the edge of the plate. We are interested in examining the relationship between upstream and downstream thickness responsible for the mechanism of elbow growth and eddy formation in terms of relevant parameters to testify the validity of certain power laws and equipartition principle.

Let us consider a control volume of width w with the paper. Thus, the jump can be treated as stationary with respect to the control volume. Assume the velocities V_1 and V_2 are uniform over the channel. By choosing the control volume to be very thin, the frictional force on the channel bed may be neglected. Let h_1 and h_2 be the heights of the fluid stream before and after jump respectively. For a bulk flow model, the density ρ may be considered constant for a small volumetric discharge Q through the control volume. Applying continuity equation we have

$$\rho w h_1 V_1 = \rho w h_2 V_2 = Q. \tag{39}$$

Hydrostatic pressure forces over each face of control volume can be accounted for momentum transfer across the faces and thus we get

$$\frac{1}{2}gh_1^2 - \frac{1}{2}gh_2^2 = V_2^2 h_2 - V_1^2 h_1. \tag{40}$$

From Eq. (39) we obtain

$$V_2 = \left(\frac{h_1}{h_2}\right)V_1. \tag{41}$$

Invoking Eq. (41) into Eq. (40) we formulate a quadratic equation in h_1 and h_2. Trivial solution of this equation leads to

$$h_1 = h_2. \tag{42}$$

Non-trivial solution is to be extracted from the following expression

$$\left(\frac{h_2}{h_1}\right)^2 + \left(\frac{h_2}{h_1}\right) - 2\left(\frac{1}{g}\right)\left(\frac{V_1^2}{h_1}\right) = 0. \tag{43}$$

Invoking $V_1 = \frac{Q}{\rho w h_1}$ from Eq. (39) into Eq. (43) we get

$$\frac{h_2}{h_1} = -\frac{1}{2} + \frac{1}{2}\sqrt{1 + (2)^2\left(\frac{1}{g\rho^2}\right)\left(\frac{Q^2}{w^2 h_1^3}\right)}. \tag{44}$$

For $h_2 \sim h_1 \sim h$ (say), we will have to have

$$(2)^2\left(\frac{1}{g\rho^2}\right)\left(\frac{Q^2}{w^2 h_1^3}\right) \sim 0. \tag{45}$$

It implies that

$$h_1 \gg (2)^{2/3}\left(\frac{1}{g}\right)^{1/3}\left(\frac{Q}{\rho w}\right)^{2/3}. \tag{46}$$

That means h_1 has a minimum of the following order

$$h_{1,\min} \sim (2)^{2/3}\left(\frac{1}{g}\right)^{1/3}\left(\frac{Q}{\rho w}\right)^{2/3}. \tag{47}$$

Thus, it can be concluded that for a fluid with definite flow geometry, depth of flow h_2 after jump scales with $\frac{2}{3}$ power of the stream volume i.e.,

$$h_{2,\min} \sim Q^{2/3}. \tag{48}$$

Next, it is interesting to recognize the results as h_1 approaches h_2; the jump becomes a small surface wave. From the energy considerations and second law of thermodynamics we confirm the fact that $V_2 < V_1$ and $h_2 > h_1$, as energy must be lost by friction through the jump. This Cauchy–Poisson problem of small amplitude wave has been studied theoretically by Rayleigh [21], Kochin [22] and Sedov [23]. Reynolds [24] performed an experimental investigation. Recently, Bejan [25] showed that the general solution of such small-amplitude wavelength in the longitudinal direction x scales with a sinusoidal function of the form

$$h(x) \sim \sin^2\left(\frac{1}{2}x\sqrt{\frac{I}{A}}\right), \tag{49}$$

where A is the cross-sectional area and I is the area moment of inertia of the stream.

As the height difference is not appreciable before and after the jump, the energy is conserved. In view of Eq. (49) flow energy is equipartitioned in the post-buckled (degenerated) stream between upper half and lower half of a sinusoid.

Now, we calculate the velocity of propagation of this small-amplitude wave. From the Eq. (39) in view of negligible jump we obtain

$$V_1 \sim V_2 \sim V \text{ (say)}. \tag{50}$$

Invoking Eq. (50) into Eq. (43) we obtain

$$V = \sqrt{gh}. \tag{51}$$

Rearranging Eq. (51) into the form

$$V = \left(\frac{1}{2}\right)^{1/2}\sqrt{2gh} \tag{52}$$

we see that velocity of flow for a negligible hydraulic jump is a scale factor $\left(\frac{1}{2}\right)^{1/2}$ of the efflux from a narrow opening at the bottom of the stream.

The maximum amplitude of the elbow is of the order [26] of $\frac{h}{2}$ and this result is confirmed by all observations of free jet flows. The post-buckled elbow region becomes a distinct eddy. If the stream (h, V) was already carrying small eddies, a large-scale turbulent structure continues to move down stream with a speed [27] on the order of $\frac{V}{2}$. This is also an instance of equipartition of velocity.

4. First geometrical construct in a shear flow

From the discussion of elemental Fermat type flow, it is evident that a stable fluid column can exist in the form of a vertical and or horizontal line segment in one-dimensional arrangement. Thus most natural choice of a fluid element in a two-dimensional static situation is in the form of a finite rectangular block. In the flow situation the geometry assumes the shape of a parallelogram. It can be guaranteed that the smallest angle θ (measured in radians and counter clockwise positive) between the two non-parallel sides of the configuration is bounded in the domain $0 \leqslant \theta \leqslant \frac{\pi}{2}$.

We consider an identified elemental area in the form of a parallelogram $ABCD$ as is in the Fig. 1(a). It is exposed to fluid pressure due to its self-weight and the force exerted by the adjacent layers in a flow situation. Its sides AB and AD are x and y respectively.

We are interested to recognize the basic geometrical shape of fluid element responsible for pressure and kinetic energy transport in a flow. We will also examine the validity of the continuum principle at every point of the flow.

Let, the thrust on the area $ABCD$ be $F(x, y)$, which is a continuous function in space variable. We complete

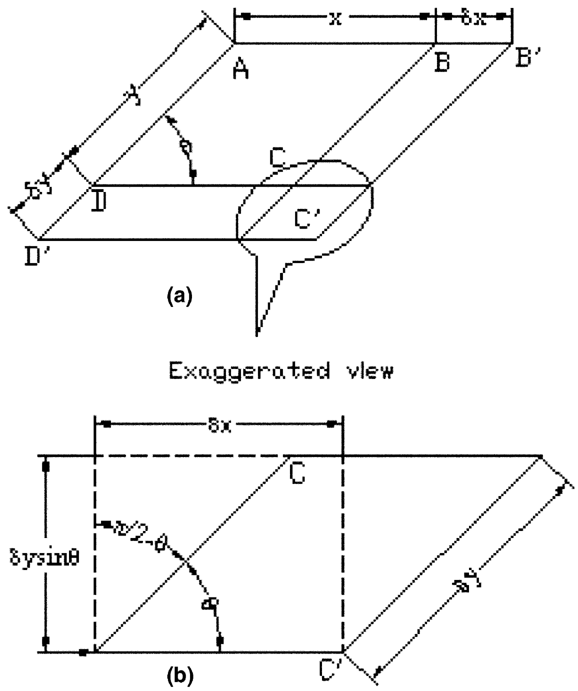

Fig. 1. First construct in a shear flow.

the parallelogram $AB'C'D'$ with sides $(x + \delta x)$ and $(y + \delta y)$. Area of the elementary parallelogram CC' is $\delta x \delta y \sin\theta$. Thrust on area CC' can be expressed as

$$F(x,y)|_{CC'} = F(x+\delta x, y+\delta y) - F(x+\delta x, y) - F(x, y+\delta y) + F(x,y). \tag{53}$$

Then, the pressure on CC', defined thrust per unit area, appears as

$$p|_{CC'} = \frac{1}{\sin\theta} \times \lim_{\delta x\to 0}\left[\frac{\underset{\delta y\to 0}{Lt}\frac{F(x+\delta x,y+\delta y)-F(x+\delta x,y)}{\delta y} - \underset{\delta y\to 0}{Lt}\frac{F(x,y+\delta y)-F(x,y)}{\delta y}}{\delta x}\right]. \tag{54}$$

Performing the sum of limits, the expression for pressure becomes

$$p|_{CC'} = \left(\frac{1}{\sin\theta}\right)\frac{\partial^2 F}{\partial x \partial y}. \tag{55}$$

If the limits were performed in a different order we would obtain

$$p|_{CC'} = \left(\frac{1}{\sin\theta}\right)\frac{\partial^2 F}{\partial y \partial x}. \tag{56}$$

Since, the thrust $F(x,y)$ is continuous in space variable, we have from Eqs. (55) and (56) the uniqueness of pressure as

$$p|_{CC'} = \left(\frac{1}{\sin\theta}\right)\frac{\partial^2 F}{\partial x \partial y} = \left(\frac{1}{\sin\theta}\right)\frac{\partial^2 F}{\partial y \partial x}. \tag{57}$$

Thus, the pressure is continuous even at the corner point, where it could be singular. Further, it is to be noted that the load bearing capacity of the fluid element is maximum when $\theta = \frac{\pi}{2}$ and it is undefined for a horizontal line element when $\theta = 0$. It is interesting to report the fact that maximum shear transport occurs when $\theta = \frac{\pi}{4}$, which is the mean value of the upper bound and lower bound of the included angle.

Next, from the flow configuration employing transformation geometry it can be conceived that rectangular shape is altered by cutting a triangular slice from the left hand side and translating it to the right hand side, for a pressure transmission from left to right. For a finite size system the elemental block could be considered comparably small to conceptualize that the shear flow takes place essentially in the form of tiny wedge packets. For pure Couette type (velocity driven) flow the interpretation is obvious. For Hagen–Poiseullie (pressure driven) flow, the situation can be thought of two super imposed Couette flows with a moving boundary at the mean line of the flow geometry. The idea expressed here has an easy extension to the local potential model [28] to the stability problems of laminar flow.

5. Conclusions

Understanding the physics of the problem can greatly simplify the mathematical calculation process of thermodynamic optimization of systems. Post analysis of the results obtained by virtue of constructal principle exhibits the property of equipartition of entities between two potentially competing forces. Thus the optimization of a thermal system, in specific, is instructive as follows. First, to choose all factors affecting system performance. An orders of magnitude analysis is to be invited to eliminate the factors not of significant contributions. In most situations two major competing forces result. They can be equated to obtain the optimum value of the parameter in concern. This design procedure is as rigorous as the method of induction in any purely mathematical prescription. The physical reality is to cast higher order constructs from smaller one by what we implement as method of induction.

Another characteristic feature of constructal theory is to predict finite shape, which is featured in the application of finite time thermodynamics. An argument from the Euclidean geometrical frame is established to predict geometric form for the first construct in a shear flow. The findings that flow proceeds in the form of wedge packet is at par with the stipulation of continuum mechanics and also conforms the observation covered in other natural flow process such as flow of radiation, information etc.

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